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TECHNICAL MEMORANDUMS  
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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No. 1018

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ON THE SYMMETRICAL POTENTIAL FLOW OF COMPRESSIBLE  
FLUID PAST A CIRCULAR CYLINDER IN THE TUNNEL  
IN THE SUBCRITICAL ZONE

By Ernst Lamla

Luftfahrtforschung  
Vol. 17, No. 10, October 26, 1940  
Verlag von R. Oldenbourg, München und Berlin

Washington  
June 1942



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ON THE SYMMETRICAL POTENTIAL FLOW OF COMPRESSIBLE  
FLUID PAST A CIRCULAR CYLINDER IN THE TUNNEL  
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The two-dimensional symmetrical potential flow of compressible fluid past a circular cylinder placed in the center line of a straight tunnel is analyzed in second approximation according to the Jansen-Rayleigh method. The departure of the profile from the exact circular shape can be kept to the same magnitude as for the incompressible flow. The velocities in the narrowest section of the tunnel wall and at the profile edge are discussed in detail.

In the treatment of the two-dimensional potential flow of an incompressible fluid past a circular cylinder placed in a tunnel with straight walls, the reflection method is resorted to; the doublet which may be envisaged in the center is reflected at the tunnel walls and the effects of all the image points added. The then obtained solution is exact; the profile washed by the air stream is, to be sure, no longer an exact circle, but the difference is ordinarily quite small. This solution for the incompressible flow is designated as the first approximation; the flow of fluid past a circular cylinder in the tunnel with allowance for the compressibility effect, discussed in the present article, forms the second approximation. The study is based on a method originally proposed by O. Janzen (reference 1) and soon afterward quoted by Lord Rayleigh (reference 2)\*\*.

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\*"Die symmetrische Potentialströmung eines Kompressibeln Gases um einen Kreiszylinder im Kanal im unterkritischen Gebiet." Luftfahrtforschung, vol. 17, no. 10, October 26, 1940, pp. 329-331.

\*\*Potential  $\phi$  and stream function  $\psi$  for the circular cylinder (and the sphere) in the free gas space have been computed up to the fourth approximation by this method for the first time by the present writer.

The considerations proceed from the stream function  $\psi$  with  $\rho$  denoting density,  $p$  pressure,  $w$  speed,  $a$  corresponding velocity of sound, and  $a_0$  the velocity of sound in the medium at rest. In absence of mass forces Euler's equations for stationary, parallel flow afford

$$\frac{1}{2} w^2 + \int \frac{dp}{\rho} = \text{const} \quad (1)$$

With the introduction of the stream function  $\psi$  the equation of continuity  $\text{div}(pw) = 0$  is satisfied by the formula

$$\left. \begin{aligned} w_x &= U \frac{\rho_\infty}{\rho} \frac{\partial \psi}{\partial y} \\ w_y &= -U \frac{\rho_\infty}{\rho} \frac{\partial \psi}{\partial x} \end{aligned} \right\} \quad (2)$$

where  $U$  is the flow velocity,  $\rho_\infty$  the density at infinity (undisturbed flow). The center line of the tunnel is the  $X$ -axis. By reason of the nonvorticity of the entire motion  $\text{rot}_z w = 0$  which, after posting (2), gives

$$\Delta \psi = \text{grad. } \psi \text{ grad } \log \rho \quad (3)$$

since ordinarily  $dp = a^2 d\rho$  it follows from (1) that

$$\frac{1}{2} \text{grad } w^2 + a^2 \text{grad } \log \rho = 0$$

whence, according to (3),

$$\Delta \psi + \frac{1}{2a^2} \text{grad } w^2 \text{ grad } \psi = 0 \quad (4)$$

If the regarded processes are adiabatic we get

$$a^2 = a_0^2 - \frac{\kappa-1}{2} w^2 \quad (5)$$

with  $\kappa = \frac{c_p}{c_v}$

Next we assume with Janzen and Rayleigh that quantity  $\psi$  can be expanded in a series, which progresses in powers of ratio  $\alpha$  from flow velocity  $U$  to sonic velocity  $a_0$ . (In this instance  $\alpha = U/a_0$  is substantially in agreement with the Mach number for undisturbed flow  $M = U/a_\infty$ .) The structure of equations (4) and (5) is such that only even powers of  $\alpha$  occur. Hence we put

$$\left. \begin{aligned} \alpha &= U/a_0 \\ \psi &= \sum_{n=0}^{\infty} \psi_n \alpha^{2n} \\ \frac{w^2}{U^2} &= K = \sum_{n=0}^{\infty} K_n \alpha^{2n} \end{aligned} \right\} \quad (6)$$

where  $\psi_n$  and  $K_n$  are no longer dependent upon  $U$  and  $a_0$ .

Replacement of  $\alpha^2$  in (4) by (5) followed by posting (6), the expansion in powers of  $\alpha$  on both sides in comparison of the coefficients of equal powers on both sides affords

$$\left. \begin{aligned} \Delta\psi_0 &= 0 \\ \Delta\psi_{n+1} &= \frac{1}{2} \sum_{m=0}^n (-\text{grad}\psi_m \text{ grad } K_{n-m} + (\kappa-1)K_{n-m} \Delta\psi_m) \end{aligned} \right\} \quad (7)$$

from which the  $\psi_n$  can be deduced consecutively. The prediction of  $K_n$  from  $\psi_n$  by means of (2) and (5) and the adiabatic equation follows at ~~from~~

$$\frac{\rho}{\rho_0} = \left[ 1 - \frac{\kappa-1}{2} \frac{w^2}{a_0^2} \right]^{\frac{1}{\kappa-1}} \quad (8)$$

Because of the appearance of density  $\rho$  the solution of the velocities from  $\psi$  is a little more complicated than that from  $\phi$ . But the use of  $\psi$  merits, because of the direct relationship with the profile form, preference in many instances. For the second approximation, that is, the prediction of  $\psi_1$  only quantity  $K_0$  is employed, for which (2), (6), and (7) afford

$$\Delta\psi_0 = 0$$

$$K_0 = \left( \frac{\partial\psi_0}{\partial x} \right)^2 + \left( \frac{\partial\psi_0}{\partial y} \right)^2 \quad \left. \right\} \quad (9)$$

$$\Delta\psi_1 = -\frac{1}{2} \left[ \frac{\partial\psi_0}{\partial x} \frac{\partial K_0}{\partial x} + \frac{\partial\psi_0}{\partial y} \frac{\partial K_0}{\partial y} \right]$$

The solutions  $\psi_0$  and  $\psi_1$  of (9) should be such that  $\psi = y$  at infinity, according to (2), that at infinity the straight lines  $y = \pm 1/2 H$  are streamlines ( $H$  = tunnel width), and lastly that streamline  $\psi = 0$  is a circle or nearly so. As solution  $\psi_0$  the one afforded by the conventional reflection method is taken: namely,

$$\psi_0 = y - \frac{R^2 \pi}{H} \frac{\sin \frac{2\pi y}{H}}{\cosh \frac{2\pi x}{H} - \cos \frac{2\pi y}{H}} \quad (10)$$

Formula (10) satisfies equation  $\Delta\psi_0 = 0$  and complies exactly with the first two of the just quoted boundary conditions. The contour  $\psi_0 = 0$  is, of course, not an exact cylinder, but the departure is slight as seen from the appended table, wherein  $r_0$ ,  $r_{45}$ , and  $r_{90}$  designate the values of radius vector  $r$  with respect to the points of the profile border incidental to  $\theta = 0^\circ$ ,  $45^\circ$ , and  $90^\circ$ .

| $H/2 R$ | $r_0/R$ | $r_{45}/r_0$ | $r_{90}/r_0$ |
|---------|---------|--------------|--------------|
| 3       | 0.9592  | 0.99868      | 0.99725      |
| 5       | .9842   | .99979       | .99960       |
| 10      | .9959   | .99999       | .99997       |
| 20      | .9990   | 1.00000      | 1.00000      |

The last two columns of the table reveal that the departure of the profile from circular form is practically insignificant. (The departure of  $r_{45}/r_0$  and  $r_{90}/r_0$  from 1 is proportional to  $(R/H)^4$ .) Even in the extreme case of the profile spanning one-third of the tunnel width the thickness of the profile is only 0.27 percent less than the width. It is to be noted that  $R$  is not the

radius of the circle as the second column indicates.

The solution of the second approximation reads  $\psi = \psi_0 + \psi_1 \alpha^2$ , and the prediction of  $\psi_1$  is predicated on the formulation of  $K_0$  according to (9) and (10).

$$K_0 = 1 - \frac{4 R^2 \pi^2}{H^2} \frac{\cosh^2 \frac{\pi x}{H} \cos \frac{2\pi y}{H} - 1}{\left( \cosh \frac{2\pi x}{H} - \cos \frac{2\pi y}{H} \right)^2} + \frac{4 R^4 \pi^4}{H^4} \frac{1}{\left( \cosh \frac{2\pi x}{H} - \cos \frac{2\pi y}{H} \right)^2} \quad (11)$$

Abbreviating these equations by means of

$$\frac{2\pi x}{H} = u; \quad \frac{2\pi y}{H} = v; \quad \frac{2\pi R}{H} = q \quad (12)$$

the last equation (9) assumes the form

$$\frac{\partial^2 \psi_1}{\partial u^2} + \frac{\partial^2 \psi_1}{\partial v^2} = \frac{H}{2\pi} \sin v \left[ \frac{q^2 \cosh u}{2 N^2} + \frac{q^4}{2 N^3} - \frac{2q^2 \sinh^2 u}{2 N^3} + \frac{q^6 \cosh u}{8 N^4} \right] \quad (13)$$

with

$$N = \cosh u - \cos v$$

Then the solution of (13) is, expressed, in analogy with (10) in the form

$$\psi_1 = - \frac{H}{2\pi} \sum_{p=1}^P \frac{g_p(u) \sin v}{N^p} \quad (14)$$

where  $N$  has the value given in (13); the  $g_p(u)$  are functions of  $u$  only, and  $P$  is a suitably chosen whole number. The advantage of this formula is as follows: It at once satisfies the boundary conditions at the tunnel wall,  $v = \pm\pi$  for  $y = \pm 1/2 H$  according to (12); hence  $\psi_1 = 0$  and  $\psi = \psi_0 = \pm 1/2 H$ . It is seen that formula (12)

is in fact practical and that  $\psi = \psi_0 = y$  for large  $x$ . Entering (14) in (13),  $H/2 \pi$  and,  $\sin v$  afford abbreviations, so that

$$\sum_{p=1}^P \left[ \frac{(p-1)^2 g_p - g_p'' + 2pg_p' \sinh u - 2p(p-1) g_p \cosh u}{N^p} \right] = \frac{q^2 \cosh u}{2 N^2} + \frac{q^4 - 2q^2 \sinh^2 u}{2 N^3} + \frac{q^6 \cosh u}{8 N^4} \quad (15)$$

(the derivations with respect to  $u$  being indicated by dashes).

Since  $v$  appears then solely in the function  $N = \cosh u - \cos v$ , a comparison of the coefficients of equal powers of  $N$  on both sides of (15) gives the consecutive differential equations for the functions  $g_p$ , and at the same time proves the possibility of formula (14). The denominator  $N'$  occurs only once in (15) (that is, on the left side for  $p = 1$ ); the correlated numerator must disappear:

$$g_1'' = 0$$

As the consideration involves the flow past a profile symmetrical with both the  $X$  and  $Y$  axis,  $\psi$  and hence all  $g_p$  in  $x$  and also in  $u$  must be even functions. Therefore:

$$g_1 = \text{const} = C_0 \quad (16a)$$

The correlated terms of  $N^2$  give

$$g_2 - g_2'' + 2 g_1' \sinh u = \frac{1}{2} q^2 \cosh u$$

or  $g_2'' - g_2 = - \frac{1}{2} q^2 \cosh u$

$$g_2 = C_1 \cosh u - \frac{1}{4} q^2 u \sinh u \quad (16b)$$

Correspondingly the further functions  $g_p$  follow at

$$g_3 = C_2 \cosh 2u + C_1 + \frac{1}{8} q^4 \quad (16c)$$

$$g_4 = C_3 \cosh 3u + \frac{3}{2} D \cosh u \quad (16d)$$

$$g_5 = C_4 \cosh 4u + (2 C_3 + D) \cosh 2u + \frac{3}{2} D \quad (16e)$$

$$g_6 = C_5 \cosh 5u + \left( \frac{5}{2} C_4 + \frac{5}{4} C_3 + \frac{5}{8} D \right) \cosh 3u + \left( \frac{5}{2} C_3 + \frac{15}{4} D \right) \cosh u \quad (16f)$$

etc. With  $D$  abbreviating for

$$D = C_2 + C_1 + \frac{1}{8} q^4 + \frac{1}{96} q^6 \quad (16g)$$

Quantities  $C_1, C_2, \dots$  etc. are integration constants. For all values  $p \geq 4$ .

$$g_{p+1}'' - p^2 g_{p+1} = 2 p g_p' \operatorname{Sinh} u - 2 p(p-1) g_p \operatorname{Cosh} u \quad (17)$$

As already stated  $\psi_1$  becomes zero for large value  $u$ ; for large  $u$  the individual summands of (14) disappear as  $e^{-u}$ , the still available integration constants are to be so defined that  $\psi_0 = 0$  disappears for the points of the original contour. The exact compliance with this requirement affords an infinite series of functions  $g_p$ ; although it can be restricted to a finite series. For instance, with the selection of  $C_3 = -3/2 D$ ,  $C_4 = -1/2 C_3 - 1/4 D = 1/2 D$ , and  $C_5 = 0$ ,  $g_6$  disappears identically, so that, according to (17) all subsequent functions  $g_7 = g_8 = \dots$  can be put = 0. Formula (14) then terminates with a series of  $P = 5$  terms with altogether three arbitrary constants  $C_0, C_1, C_2$ , which are so determined that  $\psi_1$  exactly disappears for the points of the profile of the first approximation ( $\psi_0 = 0$ ) related to  $\alpha = 0^\circ, 45^\circ$ , and  $90^\circ$ . Then the profile of the second approximation agrees at least in eight points with that of the first. Exact agreement in more points can be attained by increasing  $P$ .

In the following  $P = 3$  is deemed commensurate, the justification of this step being deferred till later. All the  $g_p$  must disappear identically for  $p \geq 4$ , which, according to (16) merely requires that  $D = 0$  and further  $C_3 = C_4 = \dots = 0$ . Substitution of two new constants

$$C_0 = c_0 q^2; \quad C_1 = c_1 q^4 \quad (18a)$$

for  $C_0$  and  $C_1$ , affords

$$C_2 = - \left( c_1 + \frac{1}{8} \right) q^4 - \frac{1}{96} q^6 \quad (18b)$$

from  $D = 0$  according to (16g), with which values formula (14) gives, if the functions  $g_p$  are taken from (16):

$$\psi_1 = -\frac{H}{2\pi} \sin v \left[ \frac{c_0 q^2}{N} + \frac{c_1 q^4 \cosh u - \frac{1}{4} q^2 u \sinh u}{N^2} + \frac{\left( c_1 + \frac{1}{8} \right) q^4 (1 - \cosh 2u) - \frac{1}{96} q^6 \cosh 2u}{N^3} \right] \quad (19)$$

Next,  $c_0$  and  $c_1$  are so defined that the square brackets in (19) exactly disappear for the points of the profile edge ( $\psi_0 = 0$ ) related to  $\theta = 0^\circ$ , and  $\theta = 90^\circ$ . For smaller values  $q$  series expansions might be used to define  $u$  and  $v$  for these points according to (10) and then  $c_0$  and  $c_1$  according to (19).

The elementary calculation gives

$$\left. \begin{aligned} c_0 &= \frac{5}{12} + \frac{5}{48} q^2 + \frac{1}{144} q^4 + \frac{1}{4320} q^6 \dots \\ c_1 &= -\frac{3}{16} - \frac{1}{72} q^2 + \frac{1}{1440} q^4 - \frac{1}{5670} q^6 \dots \end{aligned} \right\} \quad (20)$$

In equations (19) and (20)  $\psi_1$  is given. While  $c_0$  and  $c_1$  have been so defined that  $\psi_1$  becomes exactly zero for only four points of the original profile boundary, actually the thus obtained  $\psi_1$  disappears identically, as is readily proved, for all points of the original profile boundary up to the terms with  $q^4$  inclusive. The profile  $\psi_0 + \psi_1 \alpha^2 = 0$  is therefore to be regarded with the same accuracy as the circle at the profile  $\psi_0 = 0$ , which proves the validity of the choice of  $P = 3$ .

Allowing the tunnel to become wider and wider, that is, let  $q$  approach 0,  $\psi_1$  assumes the value

$$\begin{aligned} \psi_1 &= \frac{R^2}{r} \sin \theta \left( -\frac{7}{12} + \frac{1}{2} \frac{R^2}{r^2} + \frac{1}{12} \frac{R^4}{r^4} \right) \\ &\quad + \frac{R^2}{r} \sin 3\theta \left( \frac{1}{4} - \frac{1}{4} \frac{R^2}{r^2} \right) \end{aligned} \quad (21)$$

known from the theory of free flow of fluid past a circular cylinder.

The velocity is defined according to (2). Since  $\rho_\infty$  relates to  $w = U$ , equations (6) and (8), restricted to quantities of the order of  $\alpha^2$ , afford

$$\frac{\rho_\infty}{\rho} = 1 + \frac{1}{2} (K_0 - 1) \alpha^2 + \dots$$

hence (with the same degree of accuracy)

$$\left. \begin{aligned} \frac{w_x}{U} &= \frac{\partial \Psi_0}{\partial y} + \left[ \frac{\partial \Psi_0}{\partial y} \frac{K_0 - 1}{2} + \frac{\partial \Psi_1}{\partial y} \right] \alpha^2 \dots \\ \frac{w_y}{U} &= - \frac{\partial \Psi_0}{\partial x} - \left[ \frac{\partial \Psi_0}{\partial x} \frac{K_0 - 1}{2} + \frac{\partial \Psi_1}{\partial x} \right] \alpha^2 \dots \end{aligned} \right\} \quad (22)$$

Disregarding, at this instance, the general expression (22) in favor of the narrowest cross section, that is, the points  $x = 0$  or  $u = 0$ , (9) and (19) give

$$\left. \begin{aligned} (w_y)_{x=0} &= 0 \\ \left( \frac{w_x}{U} \right)_{x=0} &= 1 + \frac{q^2}{2N_0} + \left[ \left( \frac{11}{6} - \frac{7}{12} q^2 + \frac{1}{24} q^4 \right) \frac{q^2}{2N_0} \right. \\ &\quad \left. + \left( -\frac{3}{4} - \frac{q^2}{12} + \frac{q^4}{120} \right) \frac{q^4}{4N_0^2} + \frac{q^6}{96N_0^3} \right] \alpha^2 \dots \end{aligned} \right\} \quad (23)$$

with

$$N_0 = 1 - \cos v = 2 \sin^2 \frac{v}{2}$$

Two principal cases are considered.

First, the maximum velocity at the tunnel wall ( $w_K$ ), where  $y = 1/2 H$ , hence  $v = \pi$ ,  $N_0 = 2$ ; therefore

$$\frac{w_K}{U} = 1 + \frac{1}{4} q^2 + \left[ \frac{11}{24} q^2 + \frac{19}{192} q^4 \dots \right] \alpha^2 \dots \quad (24a)$$

second, the maximum speed at the profile edge ( $w_P$ ); for which (10) gives at  $\alpha = 90^\circ$

$$\text{or} \quad \frac{y}{H} = 1 - \frac{1}{24} q^2 + \frac{11}{5760} q^4 \dots$$

$$\frac{w_P}{U} = 2 + \frac{1}{6} q^2 + \frac{1}{180} q^4 \dots + \left[ \frac{7}{6} + \frac{43}{72} q^2 + \frac{47}{432} q^4 \dots \right] \alpha^2 \dots$$

(24b)

In the event that the tunnel is wide enough the terms with  $q^4$  can be disregarded relative to those with  $q^2$ , and  $R$  must then be put equal to the radius of the circle; likewise one may put  $\alpha = M$  the Mach number  $M = U/a_\infty$  in undisturbed flow without introducing great error.

Because (5) affords

$$M = \alpha \left| \sqrt{1 - \frac{\kappa - 1}{2} \alpha^2} = \alpha \left[ 1 + \frac{\kappa - 1}{4} \alpha^2 \dots \right] \right.$$

and the validity of the entire consideration holds only to about  $\alpha = 0.4$  since otherwise the maximum speed at the profile edge already reaches the value of the local velocity of sound. Hence (24a) and (24b) may be expressed in the form

$$\left. \begin{aligned} \frac{w_K}{U} &= 1 + \frac{1}{4} \left( 1 + \frac{11}{6} M^2 \right) \left( \frac{2\pi R}{H} \right)^2 \\ \frac{w_p}{U} &= 2 \left( 1 + \frac{7}{12} M^2 \right) + \frac{1}{6} \left( 1 + \frac{43}{12} M^2 \right) \left( \frac{2\pi R}{H} \right)^2 \end{aligned} \right\} \quad (25)$$

$M = 0$  furnishes the conditions in an incompressible fluid. The value 2 for the ratio  $w_p/U$  is supplemented by a value depending upon the width of the tunnel. As a consequence of the compressibility, the value 2 and the supplementary value is raised in second approximation. The essential factor now is that the percentage increase in the additional amount is substantially greater than

the rise of the basic amount 2, which is  $\frac{43}{12} M^2$  instead of  $\frac{7}{12} M^2$ , or more than six times as high.

The "correction of the flow velocity"  $\Delta U$  at the tunnel wall is given by (26a):

$$\frac{(\Delta U)_K}{U} = \frac{1}{4} \left( 1 + \frac{11}{6} M^2 \right) \left( \frac{2\pi R}{H} \right)^2 \quad (26a)$$

The value of  $\Delta U$  at the profile is, according to (25):

$$w_p = 2(U + \Delta U) \left[ 1 + \frac{7}{12} (M + \Delta M)^2 \right]$$

$w_p$  to be taken from (25).

Then

$$\frac{(\Delta U)_P}{U} = \frac{1}{12} \left( 1 + \frac{11}{6} M^2 \right) \left( \frac{2\pi R}{H} \right)^2 \quad (26b)$$

and consequently:

$$\frac{(\Delta U)_P}{(\Delta U)_K} = \frac{1}{3} \quad (27)$$

This means that no compressibility effect is perceptible in the ratio of the two  $\Delta U$  in the employed approximation.

Translation by J. Vanier,  
National Advisory Committee  
for Aeronautics.

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Page

